On the non-relativistic limit of the quantum sine-Gordon model with integrable boundary condition.

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Abstract

We show that the the generalized Calogero-Moser model with boundary potential of the Pöschl-Teller type describes the non-relativistic limit of the quantum sine-Gordon model on a half-line with Dirichlet boundary condition.

In this Letter we consider the sine-Gordon model on a half-line,

$$\mathcal{L}_{SG} = \frac{1}{2} \int_0^{+\infty} \left[(\partial_t \varphi)^2 - (\partial_x \varphi)^2 + \frac{m_0^2}{\beta^2} \cos \beta \varphi \right] dx \;+\; M \cos \frac{\beta}{2} (\varphi(x=0) - \varphi_0), \qquad (1)$$

with the fixed value of field at the boundary: $\varphi(x=0,t) = \varphi_0$, or $M = \infty$ in (1). Such a model was discussed in [1], where its quantum integrability and exact S-matrix were conjectured. The boundary scattering matrix is diagonal and, according to [1], the reflection amplitude of the soliton P_+ (resp. P_- for anti-soliton) reads:

$$P_\pm(\theta) = \cos(\xi \pm \lambda u) R(u, \xi) = \cos(\xi \pm \lambda u) R_0(u) R_1(u, \xi), \qquad (2)$$

where $\theta = iu$ is the rapidity, $\xi = \frac{4\pi}{\beta} \varphi_0$ and $\lambda = \frac{8\pi}{\beta^2} - 1$;

$$\begin{aligned}
R_0(u) &= \frac{\Gamma\left(1 - \frac{2\lambda u}{\pi}\right) \Gamma\left(\lambda + \frac{2\lambda u}{\pi}\right)}{\Gamma\left(1 + \frac{2\lambda u}{\pi}\right) \Gamma\left(\lambda - \frac{2\lambda u}{\pi}\right)} \prod_{k=1}^{\infty} \frac{\Gamma\left(4\lambda k - \frac{2\lambda u}{\pi}\right)}{\Gamma\left(4\lambda k + \frac{2\lambda u}{\pi}\right)} \times \\
&\quad \times \frac{\Gamma\left(1 + 4\lambda k - \frac{2\lambda u}{\pi}\right) \Gamma\left(\lambda(4k+1) + \frac{2\lambda u}{\pi}\right) \Gamma\left(1 + \lambda(4k-1) + \frac{2\lambda u}{\pi}\right)}{\Gamma\left(1 + 4\lambda k + \frac{2\lambda u}{\pi}\right) \Gamma\left(\lambda(4k+1) - \frac{2\lambda u}{\pi}\right) \Gamma\left(1 + \lambda(4k-1) - \frac{2\lambda u}{\pi}\right)},
\end{aligned} \qquad (3)$$

$$\begin{aligned}
R_1(u, \xi) &= \frac{1}{\pi} \prod_{l=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + 2l\lambda + \frac{-\xi + u\lambda}{\pi}\right) \Gamma\left(\frac{1}{2} + 2l\lambda + \frac{\xi + u\lambda}{\pi}\right)}{\Gamma\left(\frac{1}{2} + 2l\lambda + \lambda + \frac{-\xi + u\lambda}{\pi}\right) \Gamma\left(\frac{1}{2} + 2l\lambda + \lambda + \frac{\xi + u\lambda}{\pi}\right)} \times \\
&\quad \times \frac{\Gamma\left(\frac{1}{2} + 2l\lambda + \lambda + \frac{\xi - u\lambda}{\pi}\right) \Gamma\left(\frac{1}{2} + 2l\lambda + \lambda - \frac{\xi + u\lambda}{\pi}\right)}{\Gamma\left(\frac{1}{2} + 2l\lambda + 2\lambda + \frac{\xi - u\lambda}{\pi}\right) \Gamma\left(\frac{1}{2} + 2l\lambda + 2\lambda - \frac{\xi + u\lambda}{\pi}\right)}.
\end{aligned} \qquad (4)$$

The poles of $P_\pm$ located in the physical domain $0 < u < \pi/2$ at $u_n = \pm\frac{\xi}{\lambda} - \frac{2n+1}{2\lambda}\pi$ correspond to the "boundary" bound states of the theory. The latter exist in the soliton (resp. anti-soliton) scattering channel if $\xi > 0$ (resp. $\xi < 0$), and their energy is $E_n = M_s \cos u_n$. Note that the "physical" values of the parameter ξ are bounded [2]: $|\xi| < 4\pi^2/\beta^2$. In the semi-classical limit of the quantum field theory (1), $\beta \to 0$, the principal ("tree") approximation to the amplitudes (2) has the following form [2]:

$$P_\pm(\theta) = \exp\left(\pm i\xi + i|\xi|\right)\frac{S(\theta;0)[S(2\theta;0)]^{1/2}}{[S(\theta;\beta^2\xi/8\pi)S(\theta;-\beta^2\xi/8\pi)]^{1/2}}, \tag{5}$$

where

$$S(\theta;y) = \exp\left(\frac{8i}{\beta^2}\int_0^\theta dv \ln\tanh^2\frac{v+iy}{2}\right), \tag{6}$$

$S(\theta;0)$ being the semi-classical approximation to the bulk soliton-soliton S-matrix [3].

Our purpose here is to show that the non-relativistic dynamics of quantum sine-Gordon solitons in the presence of a boundary is described by the generalized Calogero-Moser Hamiltonian:

$$
\begin{aligned}
\hat{H} = &- \frac{1}{2M_s}\sum_{i=1}^N \frac{d^2}{dx_i^2} - \frac{1}{2M_s}\sum_{j=1}^M \frac{d^2}{dy_j^2} + \sum_{i<i'}^N \left(V_{AA}\left(x_i - x_{i'}\right) + V_{AA}\left(x_i + x_i'\right)\right) \\
&+ \sum_{j<j'}^M \left(V_{AA}\left(y_j - y_{j'}\right) + V_{AA}\left(y_j + y_{j'}\right)\right) + \sum_{i=1}^N\sum_{j=1}^M \left(V_{A\bar{A}}\left(x_i - y_j\right) + V_{A\bar{A}}\left(x_i + y_j\right)\right) \\
&\hspace{5cm} + \sum_{i=1}^N W_A\left(x_i\right) + \sum_{j=1}^M W_{\bar{A}}\left(y_j\right).
\end{aligned}
\tag{7}
$$

Here V_{AA} and $V_{A\bar{A}}$ are bulk nonrelativistic potentials obtained long ago in [3, 4]

$$V_{AA}(x) = \frac{\alpha_0^2}{M_s}\frac{\rho(\rho-1)}{\sinh^2\alpha_0 x}, \quad V_{A\bar{A}}(x) = -\frac{\alpha_0^2}{M_s}\frac{\rho(\rho-1)}{\cosh^2\alpha_0 x}, \tag{8}$$

with

$$\rho = \frac{8\pi}{\beta^2}, \tag{9}$$

$$\alpha_0 = \frac{m_0}{2}, \tag{10}$$

and W_A and $W_{\bar{A}}$ are boundary potentials of the Pöschl-Teller [5] type

$$
\begin{aligned}
W_A(x) &= \frac{\alpha_0^2}{2M_s}\left(\frac{\mu(\mu-1)}{\sinh^2\alpha_0 x} - \frac{\nu(\nu-1)}{\cosh^2\alpha_0 x}\right), \\
W_{\bar{A}}(x) &= \frac{\alpha_0^2}{2M_s}\left(\frac{\nu(\nu-1)}{\sinh^2\alpha_0 x} - \frac{\mu(\mu-1)}{\cosh^2\alpha_0 x}\right), \qquad \mu > 1, \nu > 1.
\end{aligned}
\tag{11}
$$

We will show below that μ and ν are related to the parameter ξ of the sine-Gordon model (1) as follows:

$$\frac{\nu - \mu}{2} = \frac{\xi}{\pi}. \tag{12}$$

Note that the translational invariance of the Hamiltonian (7) is broken not only by the boundary potentials, but also by the interaction of particles with their mirror images. This is very natural from the point of view of the underlying sine-Gordon theory, since it can be shown [2], that the one soliton problem on a half-line is equivalent to the three-soliton bulk problem, with one of the particles staying at $x = 0$, and the other two being "generalized mirror images" of each other (the Bäcklund transformation generalizing the method of images was constructed in [6]). The analogy becomes exact if we take $\varphi_0 = 0$. Then the ordinary method of images works, and it is obvious that the system of $N + M$ solitons on a half-line is equivalent to the system of $2(N + M)$ solitons on a line with symmetric initial conditions. Hence the corresponding nonrelativistic Hamiltonian can be obtained from the known [3, 4] nonrelativistic bulk Hamiltonian. One can easily see that the result is just (7) with $\mu = \nu$, $\mu(\mu - 1) = \rho(\rho - 1)/4$. Now let us turn to the case of arbitrary φ_0.

To establish the equivalence we will show that the S-matrices of the quantum sine-Gordon theory and the model (7) coincide in the appropriate limit. The system (7) is integrable both at the classical and quantum levels [8, 9]. To see this one takes the hyperbolic-type Calogero-Moser Hamiltonian for $N + M$ particles based on the BC_{N+M} root system [7] and shifts the coordinates of the particles $N + 1, ..., N + M$ by $i\pi/2$. The result is (7). Integrability means that the system admits a Lax representation and has $N + M$ integrals in involution. Moreover, since as $t \to \pm\infty$ these integrals reduce asymptotically to symmetric polynomials in particles' momenta, one can use the standard argument [10] to show that the S-matrix is factorized. A small modification arises due to the presence of the boundary; namely, one can consider the particles' collisions both very far from the boundary where the problem is reduced to the bulk one, and near the boundary where the colliding particles have enough time to reflect and go to $x = +\infty$. In the first case factorization gives the nonrelativistic Yang-Baxter equation for the bulk S-matrix [3], while in the second case we get exactly the boundary Yang-Baxter equation of [1], which in the Dirichlet case allows to express the boundary S-matrix of the anti-kink through that of the kink [1]. The unitarity requires the latter to be a pure phase, but otherwise leaves it undetermined. So in both theories the S-matrix is factorized and fully determined by the bulk two-particle S-matrix and the boundary S-matrix. Thus to establish the equivalence it is sufficient to show that these S-matrices coincide when the nonrelativistic limit is taken.

Let us comment first on the properties of the one-particle Schrödinger equation with the Pöschl-Teller potential $W_A(x)$. The energy of the bound states, which appear when $\nu > \mu + 1$, is given by $E_n = -\frac{\alpha_0^2}{2M_s}(\nu - \mu - 1 - 2n)^2$, where $n = 0, 1, 2...$ For a fixed value of $\nu - \mu$ there are in total $\left[\frac{\nu - \mu - 1}{2}\right]$ bound states. The reflection coefficient, which is a pure

phase, can be obtained to be equal to

$$S_A(k) = \frac{\Gamma\left(\frac{ik}{\alpha_0}\right)\Gamma\left(\frac{1}{2} + \frac{\mu-\nu}{2} - \frac{ik}{2\alpha_0}\right)\Gamma\left(\frac{\mu+\nu}{2} - \frac{ik}{2\alpha_0}\right)}{\Gamma\left(-\frac{ik}{\alpha_0}\right)\Gamma\left(\frac{1}{2} + \frac{\mu-\nu}{2} + \frac{ik}{2\alpha_0}\right)\Gamma\left(\frac{\mu+\nu}{2} + \frac{ik}{2\alpha_0}\right)} \tag{13}$$

This expression has "physical" poles on the upper imaginary half-axis in the complex momentum plane which correspond to the bound states. Besides, it has poles at the points $k_n = (1+n)\alpha_0$ that come from the first Γ-function in the numerator of (13). The latter set of poles is infinite and does not correspond to any bound states of the theory. The S-matrix for the potential $W_{\bar{A}}$ can be obtained from (13) by the substitution $\mu \leftrightarrow \nu$. One can see that S_A and $S_{\bar{A}}$ satisfy indeed the boundary Yang-Baxter equation of [1]

$$S_A \cos\left(\frac{\pi}{2}(\nu - \mu) - \lambda u\right) = S_{\bar{A}} \cos\left(\frac{\pi}{2}(\nu - \mu) + \lambda u\right). \tag{14}$$

The nonrelativistic limit of (2) corresponds to the values $\theta \ll 1$. Simultaneously we must take the limit $\beta \ll 1$, so that $M_s = \frac{8m_0}{\beta^2} \gg m_0$ (otherwise the S-matrix becomes 1 and we do not get anything interesting.) Note that this is not a quasiclassical limit, since the latter corresponds to $\frac{k}{\alpha_0} \gg 1$, where

$$k \equiv M_s\theta, \tag{15}$$

whereas according to (10) $\frac{k}{\alpha_0} = \frac{16\theta}{\beta^2}$, which is not necessarily large. We recover the quasiclassical limit in the region $\beta^2 \ll \theta \ll 1$. In what follows we assume that ξ is positive and $\beta\varphi_0 \sim 1$, so that ξ scales as $1/\beta^2$. Then P_+ has poles corresponding to the boundary bound states, and the energy of the bound states lying close to the edge of the continuous spectrum in the theory (1) becomes $E_n \simeq M_s - \frac{m_0\pi}{8\lambda}\left(\frac{2\xi}{\pi} - 1 - 2n\right)^2$. P_- does not have poles in the physical region. One can easily see then that (11),(13) describe correctly the spectrum of the boundary bound states provided that equations (10),(12) are fulfilled. To complete the identification of the boundary S-matrices we have to compare the phase shifts. Since ξ scales as $1/\beta^2$, by virtue of (12) the expression (13) can be rewritten as $S_{NR}(k, \nu - \mu)f(\theta)$, where

$$S_{NR}(k, \nu - \mu) = \frac{\Gamma\left(\frac{ik}{\alpha_0}\right)\Gamma\left(\frac{\mu-\nu}{2} - \frac{ik}{2\alpha_0}\right)}{\Gamma\left(-\frac{ik}{\alpha_0}\right)\Gamma\left(\frac{\mu-\nu}{2} + \frac{ik}{2\alpha_0}\right)} \tag{16}$$

is meromorphic and contains the poles located in the arbitrarily small neighbourhood of $\theta = 0$ as $\beta \to 0$, whereas the factor $f(\theta)$ can be expanded into the power series $f(\theta) = 1 + \sum_{l=1}^{\infty} a_l\theta^l$ with all the coefficients and a radius of convergence ~ 1 as $\beta \to 0$. In the same limit $P_{\pm}$ can be factorized analogously: $P_{\pm} = S_{NR}(k, \pm 2\xi/\pi)f_{\pm}(\theta)$ with $f_{\pm}$ admitting expansions of the form $f_{\pm}(\theta) = 1 + \sum_{l=1}^{\infty} a_l^{\pm}\theta^l$ with all the coefficients and the radius of convergence ~ 1 in the limit $\beta \to 0$. Therefore the boundary S-matrices agree when $\theta \ll 1$, and S_{NR} represents the nonrelativistic limit of the boundary S-matrix of the sine-Gordon theory. One can check that the same statements are true for the bulk two-particle sine-Gordon S-matrix of [3] and the S-matrix of the particles interacting via the potentials (8), provided that (9), (10) are satisfied. (This result was first established in [4] in the quasiclassical approximation

and later confirmed in [3]. Note that our approach allows to give an exact sense to the statement that the nonrelativistic limit of the bulk sine-Gordon theory is the hyperbolic-type Calogero-Moser model.) Thus the equivalence of (7) and the nonrelativistic limit of (1) is established.

It is instructive to compare also the combined nonrelativistic/quasiclassical limit of the S-matrix of (1) with the Pöschl-Teller S-matrix in the regime $\beta^2 \ll \theta \ll 1$ (i.e. $k \gg m_0$) without applying to the exact formula (2), similar to how it was first done in [4] for the bulk soliton scattering. This means that one should first take the limit $\beta \to 0$ and then, using (5), pass to the limit $\theta \to 0$. Expanding the integrals in (5) and using the asymptotic formulas for the Γ-functions in (13) we get for (5) and (13) in the principal order the following result:

$$P_+(k) = e^{2i\xi sign(k) + \frac{4ik}{m_0} \ln \frac{k\beta^2}{m_0}}, \qquad P_-(k) = e^{\frac{4ik}{m_0} \ln \frac{k\beta^2}{m_0}}, \tag{17}$$

once again confirming the equivalence of the two theories. Note that it is impossible to determine μ and ν separately, since in our limit the boundary S-matrices in both theories depend only on the difference $\mu - \nu$.

In conclusion we would like to mention that the identification of the nonrelativistic limit in the case of the most general integrable boundary condition [1] requires a nontrivial generalization of the Calogero-Moser Hamiltonians. Indeed, if in (1) $M < \infty$, then the boundary S-matrix does not conserve the topological charge, and we are not aware of any integrable nonrelativistic model which allows such a process.

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